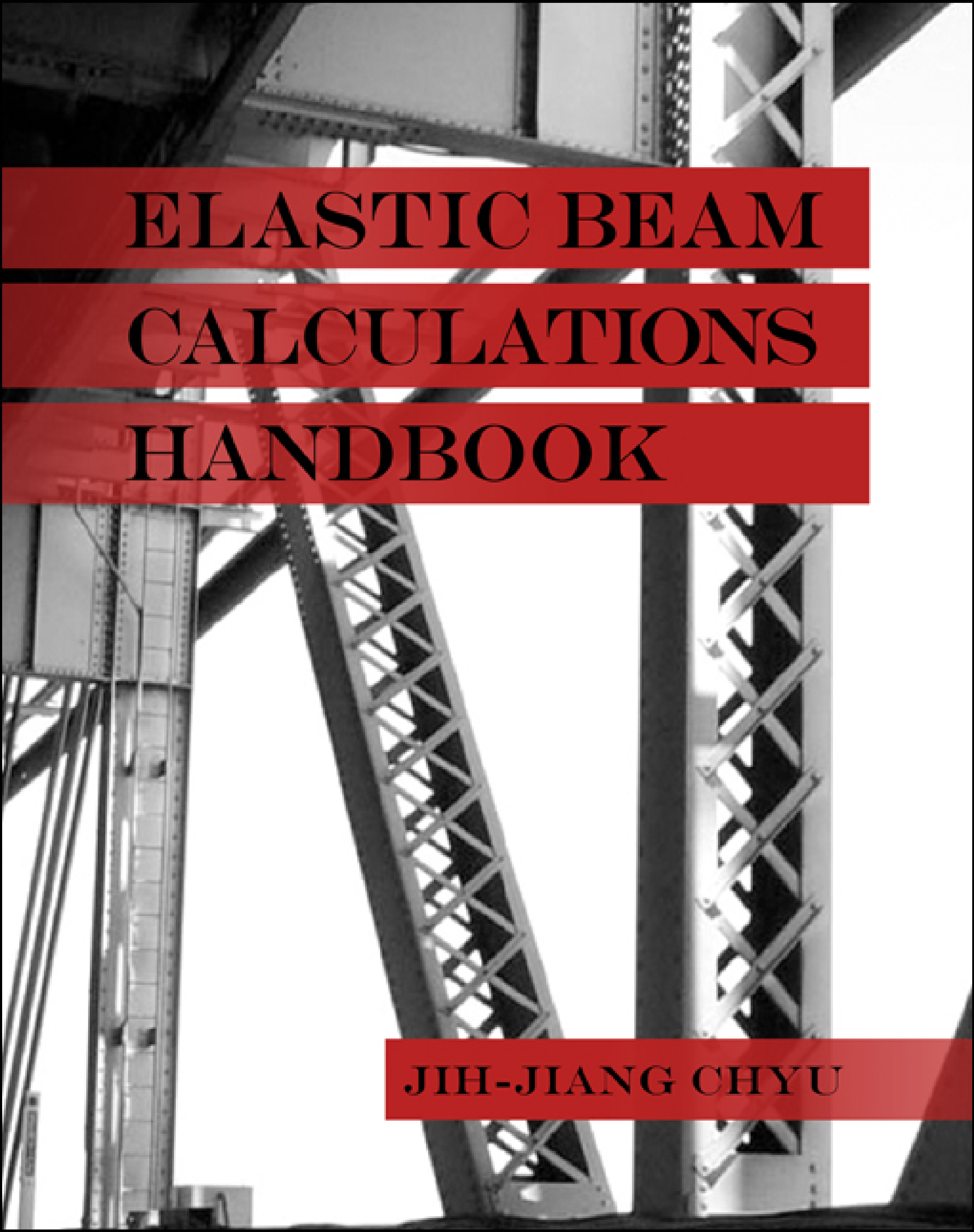
ELASTIC BEAM CALCULATIONS HANDBOOK



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4

BEAMS ON ELASTIC FOUNDATIONS

4.1. BEAMS OF INFINITE LENGTH

4.1.1. A Concentrated Force on the Beam

A concentrated force of magnitude P acts on an elastic beam on elastic foundations of infinite length, with elastic modulus k.

The deflection is given by

$$Y = CG[\cos by + \sin bx] \tag{4.1}$$

where $C = P/(8b^3J)$ is a constant and

$$G = \exp(-bx) \tag{4.2}$$

$$b = \left(\frac{k}{4J}\right)^{1/4} \tag{4.3}$$

$$J = EI_z \tag{4.4}$$

with E = Young's modulus and I_z = moment of inertia of the beam of constant cross section with respect to the horizontal principal axis z.

We want to see the effects of varying b on the deflection. Therefore, we treat Y as a function of b only while holding all other entities as constants. Thus, we take the first partial derivative of Y, denoted by Y' purely for convenience, with respect to b and obtain

$$Y' = -2CGx \sin bx + A \tag{4.5}$$

where

$$A = -3Pb^{-4}G\,\frac{\cos bx + \sin bx}{8J} \tag{4.6}$$

Note that from Expressions (4.5) and (4.6) above

$$\sin bx > 0 \quad \text{and} \quad \cos bx > 0 \tag{4.7}$$

together constitute a set of sufficient conditions for $Y' < 0$. This means that for a fixed x and for the values of b taken from the domain of definition satisfying (4.7), we have Y as a decreasing function of b. By the same token, the simultaneous satisfaction of

$$\sin bx < 0 \quad \text{and} \quad \cos bx < 0 \tag{4.7A}$$

constitutes a set of sufficient conditions for $Y' > 0$. Remarks similar to those made regarding Y as a decreasing function of b above can be made concerning Y as an increasing function of b.

We observe from the expressions for Y and Y' that both of them contain G and are proportional to G. Regarding Y, it is proportional to two additional factors, namely C and f, where f stands for ($\sin bx + \cos bx$), to which we will refer frequently below.

Now, can we say something about the sign of Y and the possibility of Y being zero?

- $Y = 0$ when $f = 0$, which means $\tan bx = 1$, or equivalently $bx = 3\pi/4 + n\pi$.
- $Y < 0$ or $Y > 0$ when $f < 0$ or $f > 0$ respectively, which means $\tan bx < 0$ or $bx > 0$ respectively. Thus, f is an indicator regarding Y being positive or negative.

It is interesting to view Y from a rather unusual perspective, namely from looking for an upper bound for Y regardless of the actual b, x values. Hence, consider f as a function of b. The function f has a maximum value of $2^{1/2}$ when $\tan bx = 1$. Thus, Y has an upper bound $2^{1/2}CG$. Therefore, f serves not only as an indicator mentioned above, but also as a scale for an upper bound for Y.

Let us examine the behavior of CG as a function of b only. We see that CG, as a decreasing function of b, has an upper bound C because $G < 1$ except when $bx = 0$, in which case $G = 1$. For C to be meaningful practically, b should be nonzero. We can see that both $2^{1/2}C$ and $2^{1/2}CG$ can serve as upper bounds for Y, but CG is a more precise upper bound for all nonzero x values. Of course, for $x = 0$, we have $CG = C$.

Expression (4.5) can be rewritten as

$$Y' = \frac{-PGN}{2k} \tag{4.8}$$

where

$$N = (2bx + 3)\ \sin\ bx + 3\cos\ bx \tag{4.9}$$

Note that from (4.8), $Y' = 0$ when $G = 0$ or $N = 0$. The former means that bx is approaching infinity, and for finite b in usual cases of practical applications this implies that the point of interest is at infinity where $Y = 0$ regardless of b, as can be seen from the expression for Y. This leads to

$$\tan bx = \frac{-3}{3 + 2bx} \tag{4.10}$$

which corresponds to $N = 0$ mentioned above and which covers two cases:

- Y has a minimum value when $\cos\ bx > 0$, with (4.10) satisfied.
- Y has a maximum value when $\cos\ bx < 0$, with (4.10) holding true.

From (4.8), we see that N determines the sign of Y':$Y' > 0$ when $N < 0$ and vice versa. Thus, from (4.9), we have $Y' < 0$ when

$$\tan bx > \frac{-3}{3 + 2bx} \tag{4.11}$$

which specifies the condition for a decreasing Y.

$Y' > 0$ when

$$\tan bx < \frac{-3}{3 + 2bx} \tag{4.12}$$

which describes the condition for an increasing Y.

Next, let us consider Expression (4.13) for the bending moment as a function of b only:

$$M = -PG\,\frac{\sin bx - \cos bx}{4b} \tag{4.13}$$

Looking at (4.13), we see that $M < 0$, $= 0$, or > 0 when $g < 0$, $= 0$, or > 0 respectively, meaning when $\tan\ bx < 0$, $= 0$, or > 0 respectively.

We have

$$M' = \frac{PGD}{4} \tag{4.14}$$

where

$$D = (ag - f) \tag{4.15}$$

with

$$a = \frac{bx + 1}{b^2} \tag{4.16}$$

$$f = \cos bx + \sin bx \tag{4.17}$$

$$g = \sin bx - \cos bx \tag{4.18}$$

Now

$$D = (a - 1) \sin bx - (a + 1) \cos bx \tag{4.19}$$

and we see that D determines not only the sign of M', but also the magnitude of M'. Moreover, $M' = 0$ when $D = 0$, and this happens when $(a - 1) \sin bx = (a + 1) \cos bx$, meaning

$$\tan bx = \frac{a + 1}{a - 1} = \frac{(1 + bx) + b^2}{1 + bx - b^2} \tag{4.20}$$

Even if $a = 1$, meaning $b = [x + (x^2 + 4)^{1/2}]/2$, we have $\cot bx = 0$, implying $bx = \pi/2 + n\pi$. Therefore, (4.20) is valid for all possible values of the entity a.

Note that the fractional function in b on the right-hand side of (4.20) is an increasing function of b. Of course, $\tan bx$ itself also is an increasing function of the variable b in the regions defined.

Note that $M' > 0$ when $D > 0$ and that $M' < 0$ when $D < 0$. Thus, M is an increasing function of b when $D > 0$ and vice versa.

4.1.2. Uniform Load on the Beam

Here the deflection is given as

$$Y = w \frac{2 - \exp(-bn) \cos bn - \exp(-bm) \cos bm}{2k} \tag{4.21}$$

where w is the uniform load intensity, and m and n are the beginning and end locations of the load along the beam. Again, we take Y as a function of b only, holding all the other entities as constants. We will use Y' to denote the first partial derivative of Y with respect to b for convenience and simplicity. Thus,

$$Y' = w \frac{A + B}{2k} \tag{4.22}$$

where

$$A = m \exp(-bm) [\cos bm + \sin bm] \tag{4.23}$$

$$B = n \exp(-bn) [\cos bn + \sin bn] \tag{4.24}$$

and k is treated as a constant even though k is involved in the definition of b. We will come back to this point later.

For now, let us note that a sufficient condition for $Y' > 0$ is the simultaneous satisfaction of the two conditions $A > 0$ and $B > 0$. This means

$$\cos bm + \sin bm > 0 \tag{4.25}$$

and

$$\cos bn + \sin bn > 0 \tag{4.26}$$

at the same time. Expressions (4.25) and (4.26) can be rewritten as

$$\tan bm > -1 \tag{4.25A}$$

$$\tan bn > -1 \tag{4.26A}$$

respectively.

Similarly, a sufficient condition for $Y' < 0$ is the simultaneous satisfaction of both $A < 0$ and $B < 0$, meaning

$$\tan bm < -1 \tag{4.27}$$

and

$$\tan bn < -1 \tag{4.28}$$

at the same time.

Of course, a sufficient condition for $Y' = 0$ is to have

$$A = B = 0 \tag{4.29}$$

leading to

$$\tan bm = \tan bn = -1 \tag{4.30}$$

The last formula may look odd, but it is a perfectly legitimate statement by noting the following fact: $\tan u = \tan v$ does not necessarily imply $u = v$. We know that bm is not equal to bn, but it is perfectly alright to have (4.30) in view of the point just made above.

It was mentioned above that we would come back to the discussion of treating k as a constant while using b as the parameter in our study. What this really means is that, in this context, b is a function of J only. Note that $db/dJ < 0$ from the definition of b given earlier in the last section. Thus,

$$[Y,J] > 0 \text{ if } [Y,b] < 0 \tag{4.31}$$

and conversely.

Also,

$$[Y,J] < 0 \text{ if } [Y,b] > 0 \tag{4.32}$$

and conversely.

Note that in the above and in the future, $[Y,u]$ is used to denote the first partial derivative of Y with respect to u for convenience. Also, we have been using ordinary derivative notations all along when we mention ahead of time that we are dealing with only one variable in that particular discourse.

Enough for the remarks so far. Let us look at things from another point of view. If we start with $b^4 = k/(4J)$, taking J as a constant and k as a function of b, then

$$[Y,k] > 0 \text{ if } [Y,b] > 0 \tag{4.33}$$

and conversely.

Also we have

$$[Y,k] < 0 \text{ if } [Y,b] < 0 \tag{4.34}$$

and conversely.

An important special case is when $m = n = mL/2$. Here

$$Y = w\,\frac{1 - \exp\left(-\dfrac{bL}{2}\right)\cos\left(\dfrac{bL}{2}\right)}{k} \tag{4.35}$$

Thus,

$$[Y,b] = \left(\frac{w}{k}\right)\left\{L\exp\left(\frac{-bL}{2}\right)\left[\cos\left(\frac{bL}{2}\right) + \sin\left(\frac{bL}{2}\right)\right]\right\} \tag{4.36}$$

We see from (4.36) that $[Y,b] > 0$ when

$$\cos\left(\frac{bL}{2}\right) + \sin\left(\frac{bL}{2}\right) > 0 \tag{4.37}$$

or

$$\tan\left(\frac{bL}{2}\right) > -1 \tag{4.37A}$$

Similarly, we have $[Y,b] < 0$ when

$$\cos\left(\frac{bL}{2}\right) + \sin\left(\frac{bL}{2}\right) < 0 \tag{4.38}$$

or

$$\tan\left(\frac{bL}{2}\right) < -1 \tag{4.38A}$$

Naturally, we have $[Y,b] = 0$ when

$$\cos\left(\frac{bL}{2}\right) + \sin\left(\frac{bL}{2}\right) = 0 \tag{4.39}$$

That is, when

$$\tan\left(\frac{bL}{2}\right) = -1 \tag{4.39A}$$

Let us check Y''. After some mathematical operations, we obtain

$$Y'' = \frac{-wL^2 \exp\left(\frac{-bL}{2}\right)\sin\left(\frac{bL}{2}\right)}{k} \tag{4.40}$$

We evaluate $\sin(bL/2)$ at the $bL/2$ value that satisfies (4.39A) for $bL/2 = 3\pi/4 + n\pi$ to reach the result

$$\sin\left(\frac{bL}{2}\right) > 0 \tag{4.41}$$

and note $\exp(-bL/2) > 0$ along with w, L, and k being all positive to conclude from (4.40) that $Y'' < 0$. Therefore, Y has a maximum here. We know that there are many values of $bL/2$ that satisfy (4.39A). As a result, there are just as many values of b that yield the same maximum value of Y for a given L. The problem can be viewed from other perspectives with individual interpretations reached. For example, we can take

$$\left(\frac{bL}{2}\right) = \frac{7\pi}{4} + n\pi \tag{4.42}$$

and obtain

$$\sin\left(\frac{bL}{2}\right) < 0 \tag{4.43}$$

while still satisfying (4.39A), thereby reaching the conclusion that we have

$$Y'' > 0 \tag{4.44}$$

which means that Y has a minimum here.

Let us consider the possibility of Y being negative or zero. We note that $\exp(-bL/2) > 1$ except when $b = 0$, in which case $\exp(-bL/2) = 1$. Immediately we see that when $b = 0$, both $\exp(-bL/2) = 1$ and $\cos(bL/2) = 1$ hold. Substituting these values into the expression for Y, we obtain $Y = 0$. But in actual situations, $b = 0$ is not likely to happen. By observation of (4.35), we know that Y is nonnegative. Also, we can reach the same conclusion by applying the result from Y' to the domain of b that is of interest to us here.

4.2. BEAMS OF SEMI-INFINITE LENGTH

4.2.1. A Concentrated Force and Moment Acting at the End of the Beam

Adopting the notation $G = \exp(-bx)$, we have the deflection as

$$Y = \left(\frac{G}{2b^3L}\right)(P\cos bx + FbM_0) \tag{4.45}$$

where

$$F = \sin bx - \cos bx \tag{4.46}$$

and P and M_0 are the applied concentrated force and bending moment respectively. If we ask whether Y can be zero or negative, the answer is yes. We have

$$Y = 0$$

when

$$P\cos bx + FbM_0 = 0 \tag{4.47}$$

In other words, Y is zero when the ratio between M_0 and P is equal to $-(\cos bx)/(bF)$, for bF being nonzero if we want to view it that way. Here we use the ratio M_0/P and not its inverse. The reason is that we like the nice dimension of "length" provided to us through M_0/P. However, we can use P/M_0 if we want to (see below).

For the case where $bx = 0$ and b is nonzero, meaning $x = 0$, (4.47) becomes

$$P + FbM_0 = 0, \quad \text{with } F = -1 \tag{4.48}$$

This means that P and M_0 are proportional to each other, with $P/M_0 = b$.

For the trivial and impractical case where $x = 0$ and $b = 0$ simultaneously, we have, from (4.47),

$$P = 0 \tag{4.49}$$

Now, what can we say about $Y > 0$ and $Y < 0$? $Y > 0$ when

$$P\cos bx + FbM_0 > 0 \tag{4.50}$$

or

$$\frac{P}{M_0} > (1 - \tan bx)b \tag{4.51}$$

if we so desire, as long as we are careful about the many kinds of behaviors of cos bx and sin bx, at different bx values, regarding positive, negative, or zero values and their implications when we come to do mathematical operations involving divisions and inequality. A naive interpretation of (4.50) is to take all the entities on the left-hand side of (4.50) as positive; then we surely automatically satisfy (4.50). When P

and M_0 are given as positive, then all it takes to do the trick is to keep both cos bx and F positive, and this is a sufficient condition for ensuring $Y > 0$.

As for $Y < 0$, the required condition is naturally

$$P \cos bx + bFM_0 < 0 \tag{4.52}$$

or

$$\frac{P}{M_0} < (1 - \tan bx)b \tag{4.53}$$

From (4.52), we see that a sufficient condition for $Y < 0$ is for both F and cos bx to be negative simultaneously, given that M_0 and P are both positive.

Furthermore, F is of critical importance, since first of all

$$F < 0 \text{ implies } 1 - \tan bx > 0 \tag{4.54}$$

$$F > 0 \text{ signifies } 1 - \tan bx < 0 \tag{4.55}$$

have far-reaching consequences as will be exemplified below. Second,

$$F = 0 \text{ means } \tan bx = 1 \tag{4.56}$$

which is a very special and important case to be explored in detail below.

Regarding $Y > 0$, the requirement is (4.51). If $F < 0$, then (4.51) may be satisfied under certain conditions. If $F > 0$, comparing (4.55) and (4.51), we see that (4.51) is unconditionally satisfied as long as $P/M_0 > 0$.

The requirement for $Y < 0$ is (4.53). We see that $F < 0$ is compatible with (4.53). When $F > 0$ is the case, then (4.53) may be satisfied if $P/M_0 < 0$.

Now let us look at the special case where $F = 0$. This means from (4.56) that

$$bx = \frac{\pi}{4} + n\pi \tag{4.57}$$

The consequences are threefold. First, Formulae (4.45), (4.47), (4.50), and (4.52) are all reduced to much simpler formulae, each of which consists of only one term, with the M_0 term missing. Second, Formula (4.50) is simplified to

$$P > 0 \tag{4.58}$$

which is free of M_0. Finally, Formula (4.52) is simplified to

$$P < 0 \tag{4.58A}$$

which is, again, independent of M_0.

Now let us look at an interesting special case, namely $x = 0$. Here

$$Y = \frac{P - bM_0}{2b^3 J} \tag{4.59}$$

Thus,

$$[Y,b] = \frac{2bM_0 - 3P}{2Jb^4} \tag{4.60}$$

from which we see that $[Y,b] < 0$ when

$$2bM_0 - 3P < 0 \tag{4.61}$$

and $[Y,b] > 0$ when

$$2bM_0 - 3P > 0 \tag{4.62}$$

Finally, in relation to the determination of extreme values of M, we find that $[Y,b] = 0$ when

$$2bM_0 - 3P = 0 \tag{4.63}$$

That is, when

$$b = \frac{3P}{2M_0} \tag{4.64}$$

for nonzero M_0.

Now we check the second derivative of Y with respect to b and find that it is positive if M_0 is positive and vice versa. Thus, for $M_0 > 0$, the value of b that satisfies (4.64) gives Y a minimum value, and of course, for $M_0 < 0$ we have a negative second derivative of Y with respect to b and hence a maximum value for Y.

Last but not least, we check $Y < 0$, $Y > 0$, and $Y = 0$ conditions. $Y = 0$ occurs when

$$P = bM_0 \tag{4.65}$$

$Y < 0$ is the case when

$$P < bM_0 \tag{4.66}$$

$Y > 0$ holds when

$$P > bM_0 \tag{4.67}$$

It is interesting to note that $b = P/M_0$ in (4.65), but $b = 3P/(2M_0)$ in (4.64).

4.2.2. Uniform Load on the Beam with a Simply Supported End

The reaction is

$$R = \frac{w}{2b} \tag{4.68}$$

which is inversely proportional to b.

Looking into the elements in b, we see that R is inversely proportional to the fourth root of k and directly proportional to the fourth root of J.

The deflection is

$$y = \left(\frac{w}{k}\right)[1 - \exp(-bx)\cos bx] \tag{4.69}$$

This expression is of exactly the same mathematical form as in a special case, specified by $m = n = L/2$, of the problem for a beam of infinite length under uniform load. Therefore, similar conclusions are expected. Thus, we have

$$[Y,b] = \left(\frac{wGx}{k}\right)[\cos bx + \sin bx] \tag{4.70}$$

from which we have $[Y,b] < 0$ when

$$\cos bx + \sin bx < 0 \tag{4.71}$$

and $[Y,b] > 0$ when

$$\cos bx + \sin bx > 0 \tag{4.72}$$

Finally, $[Y,b] = 0$ when

$$\cos bx + \sin bx = 0 \tag{4.73}$$

or

$$\tan bx = -1 \tag{4.74}$$

The second derivative of Y with respect to b evaluated at the bx that satisfies (4.74) is negative, so Y has a maximum here.

When will we have $Y = 0$? This happens when

$$\exp(-bx)\cos bx = 1 \tag{4.75}$$

There are two cases to consider.

Case 1. x = 0

As a result of this defining condition, one possibility is $\cos bx = 1$, $\exp(-bx) = 1$ simultaneously. Thus, $bx = 0, 2\pi \ldots$. The other possibility is $\exp(-bx) = 1/\cos bx$.

Case 2. x = Nonzero

We have $\exp(-bx) \neq 1$, $\exp(bx) > 1$, and $\exp(-bx) < 1$. We need $\cos bx > 1$ to satisfy (4.75). But this is not possible because of the property of cosine function. Therefore, this case does not exist. In other words, this case does not have physical meaning.

Next, when will we have $Y > 0$? This happens when

$$1 > \exp(-bx)\ \cos\ bx \tag{4.76}$$

meaning

$$\frac{1}{G} > C, \quad \text{with } C = \cos bx,\ G = \exp(-bx) \tag{4.77}$$

But

$$G > 1 \tag{4.78}$$

in general and

$$G = 1 \tag{4.79}$$

only when $bx = 0$, which is excluded from (4.76). Thus, (4.76) or (4.77) is established, meaning $C < 1$, as the defining condition for $Y > 0$.

What about $Y < 0$? This requires

$$1 < GC \tag{4.80}$$

meaning

$$\frac{1}{G} < C \tag{4.81}$$

But this is not possible, for two reasons. First, we have $1/G > 1$, except $bx = 0$, in which case we have $Y = 0$ already and is, therefore, excluded from our consideration here. Second, we know that C can never be greater than 1. Then how can C be greater than $1/G$? Therefore, (4.80) is not satisfied, and it is not possible to have $Y < 0$.

4.2.3. Uniform Load on the Beam with a Fixed End

It is interesting to note that the reaction here is twice that in the case of a simply supported end. That is,

$$R = \frac{w}{B} \tag{4.82}$$

Naturally, it has all the characteristics described in the last section about R.

Let us look at the bending moment

$$M = \frac{-2b^2Jw}{k} \tag{4.83}$$

which is negative.

Since b, k, and J are related via

$$b = \left[\frac{k}{4J}\right]^{1/4} \tag{4.84}$$

we may rewrite (4.83) as

$$M = -w\left(\frac{J}{k}\right)^{1/2} \tag{4.85}$$

Thus, we observe from (4.85) that in addition to being directly proportional to the uniform load intensity w, the bending moment M is directly proportional to the square root of J and inversely proportional to the square root of k.

We see also from the definition of b in terms of J, k that there are many ways to go about varying b, depending on our goals and plans. For example, we may keep J constant and vary k systematically, do the opposite, or even change both J and k simultaneously in a specific manner of choice. This can be done numerically using a computer or can be pursued further analytically to suit individual needs. This is just an example. There are many other problems that can be approached in this way, and the procedure outlines and methodology have been indicated previously.

Let us come back to the study of M. For keeping k as a constant, we have

$$M = \frac{-w}{2b^2} \tag{4.86}$$

which is inversely proportional to the square of b. For letting k vary with b via

$$k = 4Jb^2 \tag{4.87}$$

we have also the expression for M as shown in (4.86).

For our study of beams on elastic foundations, most of the subjects are considered as functions of one independent variable b, while all other entities are viewed as constants, including the longitudinal axis x. However, we know from the definition of b that b is dependent on two other entities, J and k. This point was touched upon above and will be elaborated on a little more now.

In order to assess the effects of the independent variables J and k on a subject under study, we may, first of all, obtain the results from considering the subject as a function of b only, as we did for most of the topics earlier. Then we may consider b as function of two variables J and k and obtain first and second partial derivatives of b with respect to J and k separately. Finally, we use the principles of calculus, including the chain rule, to obtain all the necessary partial derivatives of the subject function with respect to J and k, including the so-called mixed second partial derivatives, and evaluate the possible maximum and minimum values of subject functions using the information thus collected.

Based on the remarks just presented, the following is provided. Considering b as a function of two independent variables, J and k, we have the first partial derivative of b with respect to J as

$$[b,J] = \left(-\frac{1}{4}\right)\left(\frac{k}{4}\right)^{1/4} J^{-5/4} \tag{4.88}$$

which is always negative (never zero or positive), signifying b as a decreasing function of J.

The second partial derivative of b with respect to J is

$$[b,JJ] = \left(\frac{5}{16}\right)\left(\frac{k}{4}\right)^{1/4} J^{-9/4} \tag{4.89}$$

which is positive.

Now, we have

$$[b,k] = A(k)^{-3/4} \tag{4.90}$$

where

$$A = \frac{1}{4(4J)^{1/4}} \tag{4.91}$$

Note that $[b,k]$ is positive, and its value decreases as J increases and/or k increases, although at different rates, with the latter being the higher one.

Next we have

$$[b,kk] = \left(\frac{-3}{4}\right)A(k)^{-7/4} \tag{4.92}$$

which is negative, and its "absolute" value decreases as either J or k increases, with k being the stronger player again.

Finally, a so-called mixed second partial derivative of b is

$$[b,Jk] = \left(-\frac{1}{4}\right)(4J)^{-5/4}k^{-3/4} \tag{4.93}$$

which is negative also.

From (4.90) and (4.93), it is interesting to see that

$$[b,Jk] = -\frac{[b,k]}{4J} \tag{4.94}$$

Of course, we can also rewrite (4.93) as

$$[b,Jk] = \left(\frac{-1}{64}\right)(J)^{-5/4}\left(\frac{k}{4}\right)^{-3/4} \tag{4.95}$$

Thus, we have another interesting result:

$$[b,Jk] = \frac{[b,J]}{4k} \tag{4.96}$$

Note that here we have

$$[b,Jk] = [b,kJ] \tag{4.97}$$